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Field-Theoretical Space-Time Quantisation

E. PAPP

Polytechnic Institute of Cluj, Physics Department, Cluj, Romania

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Abstract

Using the framework of the quantum field theory it is proved that the possibility exists of performing—within binary description formalism—a suitable space-time quantisation. In this respect the mutual connection between the action quantisation and the space-time quantisation has been established.

1. *Introduction*

Within the framework of the quantum field theory attempts have been made to define space-time position states and operators (Schröder, 1964; Broyles, 1970). However, up to now no consistent approach to the field-theoretical space-time quantisation problem has been explicitly formulated. In this respect we shall prove that the possibility exists of performing a suitable field-theoretical space-time binary description. This description is able to support the existence of the binary space-time entities previously analysed (Papp, 1971–1973). The quantum character of the above-mentioned description will also be stated, thus pointing out the mutual connection between the requirements of the action quantisation and the 'objective' existence of the non-zero time imprecision. For this purpose the usual meaning of the action quantisation, consisting of the statement of Planck's constant as the minimum action encountered in nature, will be extended. In order to define the time operators, certain boundary conditions are implied. In these conditions the present binary time operators and also the hermiticity property will be mathematically defined only with respect to a subspace of the whole Hilbert space. As in previous papers (Papp, 1971–1973) such formalism is not to be considered as a rejection, but as an extension of Neumann's axiom (von Neumann, 1932).

Using the configuration space method and assuming in agreement with Wightmann & Schweber (1959) and Schröder (1964), the validity of the simple particle interpretation, the binary space-time operators of the non-relativistic Schrödinger (S)-field and of the relativistic Klein–Gordon

(K–G)-field are defined within the vector space description in Sections 2 and 4 respectively. The space-time compatibility is analysed in Section 3. In Section 5 it is shown that—at a given value of the angular momentum (and especially at $l=0$)—the radial space operator and the corresponding time operator may be adequately defined, as in the quantum-mechanical case, only at large t-values. The averages of the above field-theoretical space-time operators turn out to be identical with the usual quantum-mechanical space-time averages. Finally the mutual connection between action quantisation and the space-time quantisation is analysed in Section 6.

We shall work with the units for which $\hbar = c = 1$ and we shall consider, for simplicity, only the single particle states of the S- and K–G-field.

2. *The Space-Time Operators of the S-Field*

As in the quantum-mechanical case we shall define the space and time operators by means of their canonical conjugacy with momentum and energy respectively. With these conditions defining the space operator as

$$\mathbf{X}(t) = \int d\mathbf{x}\, \psi^*(\mathbf{x}, t)\, \mathbf{x}\, \psi(\mathbf{x}, t) \tag{2.1}$$

where the S-field operator is given by

$$\psi(\mathbf{x}, t) = (2\pi)^{-3/2} \int d\mathbf{p}\, a(\mathbf{p}) \exp i(\mathbf{p}\cdot\mathbf{x} - \omega t), \qquad \omega = \frac{p^2}{2m} \tag{2.2}$$

we obtain

$$\mathbf{X}(t) = \int d\mathbf{p}\, a^*(\mathbf{p}) \left[i\frac{\partial}{\partial \mathbf{p}} + t\frac{\mathbf{p}}{m_0} \right] a(\mathbf{p}) \tag{2.3}$$

The above vectorial space operator represents a set of three reciprocally commuting coordinate space operators $\{X_i(t)\}$. These components are conjugated to the corresponding components of the momentum operator. The hermiticity of the space operator $\mathbf{X}(t)$ is assured as soon as the operator $a(\mathbf{p})$ vanishes at infinity.

We shall consider that the above condition is generally fulfilled. Within the quantum field theory the canonical conjugacy preserves quantum-mechanical sense only with respect to the single particle states. Conversely, effective space and time operators per particle have to be defined. In the latter case the field-theoretical space-time description corresponds to the quantum-mechanical space-time description of a system of free identical particles.

If the classical time of the evolution along the x_1-axis is given by x_1/v_1, in the quantum-mechanical case we may consider—by virtue of the correspondence principle—the operators $x_1\hat{v}_1^{-1}$, $\hat{v}_1^{-1}x_1$ and respectively $\frac{1}{2}(x_1\hat{v}_1^{-1} + \hat{v}_1^{-1}x_1)$, where the velocity operator is given by

$$\hat{v}_1 = \frac{1}{m_0}\hat{p}_1 = -\frac{i}{m_0}\frac{\partial}{\partial x_1} \tag{2.4}$$

so that

$$\hat{v}_1^{-1} \exp i\mathbf{p}\cdot\mathbf{x} = \left(\frac{p_1}{m_0}\right)^{-1} \exp i\mathbf{p}\cdot\mathbf{x} \tag{2.5}$$

Without performing a detailed mathematical analysis of the existence of the operator $\hat{v}_1^{-1} \equiv v_1^{-1}$, which was rather formally used, we can simply remark that in this way the existence of a singularity at $p_1 = 0$ is implied. Performing the calculations and assuming that

$$\lim_{p_1\to 0} p_1^{-1} \int dp_2\, dp_3\, a^*(\mathbf{p})\, a(\mathbf{p}) < \infty \tag{2.6}$$

where the limit has to be considered in the weak sense, it may be proved that the field-theoretical counterparts of the above time operators are given by

$$T_1(t) = \int d\mathbf{p}\; a^*(\mathbf{p}) \left[i\frac{m_0}{p_1}\frac{\partial}{\partial p_1} + t - i\frac{m_0}{p_1^{\,2}} \right] a(\mathbf{p}) \tag{2.7}$$

$$T_1{}'(t) \equiv T_1{}^*(t) = \int d\mathbf{p}\, a^*(\mathbf{p}) \left[i\frac{m_0}{p_1}\frac{\partial}{\partial p_1} + t \right] a(\mathbf{p}) \tag{2.8}$$

and the hermitian operator

$$T_1^{(*)}(t) = \int d\mathbf{p}\, a^*(\mathbf{p}) \left[i\frac{m_0}{p_1}\frac{\partial}{\partial p_1} + t - i\frac{m_0}{2p_1^{\,2}} \right] a(\mathbf{p}) \tag{2.9}$$

respectively. The boundary conditions (2.5)—which state mathematically the time description only with respect to a Hilbert subspace—are needed in order to remove the singularity previously mentioned. With this condition the consistency requirement of the mathematical description is assured. The existence of the two quantum mechanical time operators $T_1(t)$ and $T_1{}'(t) \neq T_1(t)$ is in fact an expression of the canonical space-momentum conjugation, i.e. of the impossibility of precisely measuring both space and momentum. One can easily ascertain that the time operators $T_1(t)$, $(T_1{}^*(t))$ may be obtained by adding to (subtracting from) the hermitian time operator the operator:

$$iI_1 = \frac{i}{2} \int d\mathbf{p}\, \frac{m_0}{p_1^{\,2}}\, a^*(\mathbf{p})\, a(\mathbf{p}) \tag{2.10}$$

Because the hermitian part of the binary time operators $T_1(t)$ and $T_1{}^*(t)$ is identical with the hermitian time operator $T_1^{(*)}(t)$, it may be concluded that the operator T_1 plays the role of the imprecision operator of the time measurement. One may therefore infer that the quantum-mechanical time cannot be measured without an 'objective' non-zero time imprecision. The existence of this imprecision explains the existence and the meaning of the binary time operator. Consequently, the real and imaginary part of the $T_1(t)$ or $T_1{}^*(t)$ average may be defined as 'observable' evaluation and imprecision of the time measurement, respectively. In this situation we can affirm that the time quantisation consists essentially of the definition and interpretation of the binary time operator.

Obtaining the average value of the binary time operator $T_1(t)$ with respect to the single particle state

$$|\psi_1\rangle = \int d\mathbf{p}\, g(\mathbf{p})\, a^*(\mathbf{p})|0\rangle \tag{2.11}$$

it may easily be proved that the imprecision previously introduced of the time measurement is given by

$$\langle\psi_1|I_1|\psi_1\rangle = \left\langle\frac{m_0}{2p_1{}^2}\right\rangle = \int d\mathbf{p}\, \frac{m_0}{2p_1{}^2}|g(\mathbf{p})|^2 \tag{2.12}$$

where the state $|\psi_1\rangle$ has been normalised to unity. The imprecision so obtained agrees with the one previously obtained in the quantum mechanical case (Papp, 1971).

The field-theoretical counterparts of the free evolution times along the x_2- and x_3-coordinate axis may be similarly evaluated as soon as

$$\lim_{p_2\to 0} p_2^{-1} \int dp_1\, dp_3\, a^*(\mathbf{p})\, a(\mathbf{p}) < \infty \tag{2.13}$$

and respectively

$$\lim_{p_3\to 0} p_3^{-1} \int dp_1\, dp_2\, a^*(\mathbf{p})\, a(\mathbf{p}) < \infty \tag{2.14}$$

The time operators so obtained commute so that the above 'three-dimensional' time entity $\{T_i(t), i = 1,2,3\}$ may be reduced to a one-dimensional entity. For this purpose the measuring apparatus, which includes (in agreement with Broyles (1970)) the reference frame, has to be chosen so that the direction of the free motion becomes identical to that, e.g., of the x_1-axis.

Obtaining the average value of expressions (2.6), (2.13) and (2.14) with respect to the state $|\psi_1\rangle$, one gets the boundary conditions

$$\lim_{p_i\to 0} p_i^{-1} \int dp_j\, dp_k |g(\mathbf{p})|^2 < \infty, \qquad i \neq j \neq k \tag{2.15}$$

Without affecting generality, we may consider that the above boundary conditions are satisfied only when—up to a phase factor—the factorisation property

$$g(\mathbf{p}) = g_1(p_1)\, g_2(p_2)\, g_3(p_3) \tag{2.16}$$

is assumed, where

$$\lim_{p_i\to 0} p_i^{-\frac{1}{2}} g_i(p_i) < \infty, \qquad i = 1, 2, 3 \tag{2.17}$$

The above factorisation property signifies that the field-theoretical counterparts of the free evolution times along the x_1-, x_2- and x_3-axis cannot be defined without assuming the mutual independence property of the corresponding free evolution processes. Consequently, the factorisation property, which expresses in fact an initial condition needed to perform the present quantum-mechanical time description, may be re-obtained (as one would expect) in terms of the consistency conditions of the proposed mathematical description.

Starting from the binary time operator $T_1(t)$, we may subsequently define the binary space operator corresponding to the x_1-coordinate as

$$X_1^{(b)}(t) = \int d\mathbf{p}\, a^*(\mathbf{p}) \left[i\frac{\partial}{\partial p_1} + t\frac{p_1}{m_0} - \frac{i}{2} \right] a(\mathbf{p}) \tag{2.18}$$

Contrary to the time imprecision $\langle m_0/2p_1{}^2 \rangle$, which always takes non-zero values, there are cases in which the space imprecision $\langle 1/2p_1 \rangle$ vanishes—e.g. when $g(\mathbf{p})$ is either an even or an odd function of the variable p_1. Therefore the binary space-time description does not generally take a symmetric form with respect to space and time.

3. *Space-Time Compatibility*

In order to assure the possibility of the space-time description previously performed fulfilment of the space-time compatibility requirement is needed. Indeed time has been defined with respect to space. We shall formulate the binary space-time compatibility conditions requiring in agreement with the binary interpretation formalism (Kálnay & Toledo, 1967; Papp, 1973), that the average of the commutator does not possess (a well-defined or an 'undetermined') measurable meaning. The binary space-time compatibility refers not only to the pairs of binary operators but also the mixed hermitian–binary pairs.

Thus, in the one-dimensional case, the compatibility between binary space and binary time is assured under the conditions

$$|\mathrm{Re}\langle\psi_1|[X_1^{(b)}(t), T_1(t)]|\psi_1\rangle| \leqslant |\mathrm{Im}\langle\psi_1|[X_1^{(b)}(t), T_1(t)]|\psi_1\rangle \tag{3.1}$$

where account has been taken of the peculiarities of the binary description formalism. As

$$[X_1^{(b)}(t), T_1(t)] = \int d\mathbf{p}\, a^*(\mathbf{p}) \left[\frac{m_0}{p_1}\frac{\partial}{\partial p_1} - \frac{3}{2}\frac{m_0}{p_1{}^2} - i\frac{t}{p_1} \right] a(\mathbf{p}) \tag{3.2}$$

the condition (3.1) becomes

$$m_0 \left| \left\langle \frac{1}{2p_1{}^3} \right\rangle \right| \leqslant \left| \left\langle \frac{m_0}{p_1{}^2}\frac{\partial}{\partial p_1} \arg g(\mathbf{p}) - \frac{t}{p_1} \right\rangle \right| \tag{3.3}$$

We notice that space-time compatibility is particularly assured when $\langle 1/2p_1{}^3 \rangle = 0$. But considering the general case, and assuming for simplicity that the function $(1/p_1{}^2)(\partial/\partial p_1)\arg g(\mathbf{p})$ covers the relatively narrow domain in which $|g(\mathbf{p})|^2$ takes appreciable values, condition (3.3) becomes, within a certain approximation,

$$m_0 \left\langle \frac{1}{2p_1{}^2} \right\rangle \leqslant \left| \left\langle \frac{m_0}{p_1}\frac{\partial}{\partial p_1} \arg g(\mathbf{p}) - t \right\rangle \right| \tag{3.4}$$

where the factor $1/p_1$ has been extracted. It may be easily proved that the inequality (3.4) is fulfilled as soon as either

$$t \leqslant m_0 \left\langle \frac{1}{p_1} \frac{\partial}{\partial p_1} \arg g(\mathbf{p}) \right\rangle - \left\langle \frac{m_0}{2p_1^2} \right\rangle \tag{3.5}$$

or

$$t \geqslant m_0 \left\langle \frac{1}{p_1} \frac{\partial}{\partial p_1} \arg g(\mathbf{p}) \right\rangle + \left\langle \frac{m_0}{2p_1^2} \right\rangle \tag{3.6}$$

take place. Consequently, substituting the macroscopic time parameter t by the time interval

$$\left[t - \left\langle \frac{m_0}{2p_1^2} \right\rangle, t + \left\langle \frac{m_0}{2p_1^2} \right\rangle \right] \tag{3.7}$$

inequality (3.4) is satisfied. Consequently the compatibility of the binary space and time is assured as soon as time gets a binary meaning, with the imprecision given by $\langle m_0/2p_1^2 \rangle$. Setting $t = 0$ it may be easily proved, by virtue of relation (3.4), that space-time compatibility is also assured when one confers measurable meaning on the time-shift evaluation

$$\left\langle \frac{m_0}{p_1} \frac{\partial}{\partial p_1} \arg g(\mathbf{p}) \right\rangle$$

It may be proved that the compatibility between binary space $X_1^{(b)}(t)$ and hermitian time $T_1^{(*)}(t)$ may be satisfied under the same conditions as before. The compatibility of the hermitian space operator $X(t)$ with the binary time requires a doubly larger imprecision than the previous one, namely the time imprecision $\langle m_0/p_1^2 \rangle$. But introducing the concept of extended binary 'equivalence', the binary evaluations with the imprecision $\langle m_0/p_1^2 \rangle$ may be included in the same 'equivalence' class as the previous ones.

It is worthwhile mentioning that the time imprecision introduced in the above-mentioned manner is identical with the imprecision operator average. This last fact constitutes an expression of the inner consistency of the binary space-time description.

4. *The Space-Time Description of the K–G Field*

In the relativistic case there are some additional difficulties arising from the relatively more complicated functional dependence between energy, momentum and velocity. Defining, in agreement with Schröder (1964), the space operator as

$$\mathbf{X}(t) = i \int d\mathbf{x}\, \Phi^{(+)*}(\mathbf{x}, t) \overleftrightarrow{\partial_t}\, \mathbf{x}\, \Phi^{(+)}(\mathbf{x}, t) \tag{4.1}$$

where the K–G-field operator is given by

$$\Phi^{(+)}(\mathbf{x}, t) = (2\pi)^{-3/2} \int \frac{d\mathbf{p}}{\sqrt{(2p_0)}} a^{(+)}(\mathbf{p}) \exp i(\mathbf{p}\cdot\mathbf{x} - p_0 t),$$
$$p_0 = \sqrt{(\mathbf{p}^2 + m_0^2)} \tag{4.2}$$

one obtains the hermitian operator

$$\mathbf{X}(t) = \int d\mathbf{p}\, a^{(+)*}(\mathbf{p}) \left[i\frac{\partial}{\partial \mathbf{p}} + t\frac{\mathbf{p}}{p_0} \right] a^{(+)}(\mathbf{p}) \tag{4.3}$$

when the annihilation operator $a^{(+)}(\mathbf{p})$ vanishes at infinity. The components of the above vector space operator commute so that the unequivocality of the space description is assured. However, the situation is more complicated in the case of the time description.

Calculating the field-theoretical counterparts

$$T_1(t) = i \int d\mathbf{x}\, \Phi^{(+)*}(\mathbf{x}, t)\, \overleftrightarrow{\partial}_t\, x_1 \hat{v}_1^{-1}\, \Phi^{(+)}(\mathbf{x}, t) \tag{4.4}$$

and

$$T_1'(t) = i \int d\mathbf{x}\, \Phi^{(+)*}(\mathbf{x}, t)\, \overleftrightarrow{\partial}_t\, \hat{v}_1^{-1} x_1\, \Phi^{(+)}(\mathbf{x}, t) \tag{4.5}$$

of the quantum-mechanical time operators $x_1\hat{v}_1^{-1}$ and $\hat{v}_1^{-1}x_1$, where the velocity operator takes the form

$$\hat{v}_1 = -\left(\frac{\partial}{\partial x_1}\right)\left(\frac{\partial}{\partial t}\right)^{-1} \tag{4.6}$$

we obtain, assuming the validity of the boundary condition

$$\lim_{p_1 \to 0} p_1^{-1} \int dp_2\, dp_3\, p_0\, a^{(+)*}(\mathbf{p})\, a^{(+)}(\mathbf{p}) < \infty \tag{4.7}$$

(where as previously the limit has to be considered in the weak sense), the results:

$$T_1(t) = \int d\mathbf{p}\, a^{(+)*}(\mathbf{p}) \left[i\frac{p_0}{p_1}\frac{\partial}{\partial p_1} + t - i\frac{m_0^2 + p_2^2 + p_3^2}{p_0 p_1^2} \right] a^{(+)}(\mathbf{p}) \tag{4.8}$$

and

$$T_1'(t) \equiv T_1^*(t) = \int d\mathbf{p}\, a^{(+)*}(\mathbf{p}) \left[i\frac{p_0}{p_1}\frac{\partial}{\partial p_1} + t \right] a^{(+)}(\mathbf{p}) \tag{4.9}$$

respectively. Therefore, the hermitian time operator is given by

$$T_1^{(*)}(t) = \int d\mathbf{p}\, a^{(+)*}(\mathbf{p}) \left[i\frac{p_0}{p_1}\frac{\partial}{\partial p_1} + t - i\frac{m_0^2 + p_2^2 + p_3^2}{2p_0 p_1^2} \right] a^{(+)}(\mathbf{p}) \tag{4.10}$$

when condition (4.7) is fulfilled.

The time operators $T_2(t)$ and $T_3(t)$ corresponding to the free evolutions along the x_2- and x_3-axis may be similarly evaluated. Contrary to the non-relativistic case the time operators $T_1(t)$, $T_2(t)$ and $T_3(t)$ do not commute. Indeed

$$[T_1(t), T_2(t)] = \int d\mathbf{p}\, a^{(+)*}(\mathbf{p}) \left[\frac{1}{p_1}\frac{\partial}{\partial p_1} - \frac{1}{p_2}\frac{\partial}{\partial p_2} + \frac{1}{p_2^2} - \frac{1}{p_1^2} \right] a^{(+)}(\mathbf{p}) \tag{4.11}$$

and a similar result follows in the other cases. Averaging the above commutator with respect to the single particle state

$$|\Phi_1^{(+)}\rangle = \int d\mathbf{p}\, g^{(+)}(\mathbf{p})\, a^{(+)*}(\mathbf{p})|0\rangle \tag{4.12}$$

where $\langle\Phi_1^{(+)}|\Phi_1^{(+)}\rangle = 1$, we obtain the result

$$\langle\Phi_1^{(+)}|[T_1(t), T_2(t)]|\Phi_1^{(+)}\rangle = \frac{1}{2}\left\langle\frac{1}{p_2^2} - \frac{1}{p_1^2}\right\rangle + i\left\langle\left(\frac{1}{p_1}\frac{\partial}{\partial p_1} - \frac{1}{p_2}\frac{\partial}{\partial p_2}\right)\arg g^{(+)}(\mathbf{p})\right\rangle \tag{4.13}$$

The commutation condition of the time operators is thus assured (in the weak sense) if

$$\left\langle\frac{1}{p_1^2}\right\rangle = \left\langle\frac{1}{p_2^2}\right\rangle = \left\langle\frac{1}{p_3^2}\right\rangle \tag{4.14}$$

and also

$$\left\langle\frac{1}{p_1}\frac{\partial}{\partial p_1}\arg g^{(+)}(\mathbf{p})\right\rangle = \left\langle\frac{1}{p_2}\frac{\partial}{\partial p_2}\arg g^{(+)}(\mathbf{p})\right\rangle = \left\langle\frac{1}{p_3}\frac{\partial}{\partial p_3}\arg g^{(+)}(\mathbf{p})\right\rangle \tag{4.15}$$

where the single particle amplitude obeys the boundary conditions

$$\lim_{p_i\to 0} p_i^{-1}\int dp_j\, dp_k |g^{(+)}(\mathbf{p})|^2 < \infty, \qquad i \neq j \neq k \tag{4.16}$$

But in order to fulfil conditions (4.14)–(4.16) we have to impose the factorisation property

$$g^{(+)}(\mathbf{p}) = g_1^{(+)}(p_1)\, g_2^{(+)}(p_2)\, g_3^{(+)}(p_3) \tag{4.17}$$

where

$$g_1^{(+)} = g_2^{(+)} = g_3^{(+)} \tag{4.18}$$

and where

$$\lim_{p_i\to 0} p_i^{-\frac{1}{2}} g_i^{(+)}(p_i) < \infty, \qquad i = 1, 2, 3 \tag{4.19}$$

Consequently the above time description of the relativistic K–G-field may be unequivocally defined only in the one-dimensional case. Indeed the three-dimensional vectorial description possesses—by virtue of conditions (4.17)–(4.18)—a purely formal meaning and reduces effectively to a one-dimensional description. In this respect the time description of the relativistic K–G-field is, in the main, more restrictive than the case of the non-relativistic field.

Neglecting condition (4.15), relations (4.17) and (4.18) remain valid only up to a phase factor. This is the situation when the condition of binary time compatibility is imposed. Indeed, the binary time compatibility is always fulfilled as soon as the relations (4.14) are satisfied as well as relations (4.16). In this respect, excepting non-significant situations, we may consider that the binary time compatibility condition is essentially expressed by means of relation (4.14) and vice versa.

The present relativistic one-dimensionality requirement becomes manifest in the case of the Dirac electron-field. Performing the evaluation of the space operator

$$\mathbf{X}(t) = \int d\mathbf{x}\, \chi^*(\mathbf{x}, t)\, \mathbf{x}\, \chi(\mathbf{x}, t) \tag{4.20}$$

where

$$\chi(\mathbf{x}, t) = (2\pi)^{-3/2} \sum_s \int d\mathbf{p} \sqrt{\frac{m_0}{p_0}}\, u^{(+)}(\mathbf{p}, s)\, b^{(+)}(\mathbf{p}, s) \exp i(\mathbf{p}\cdot x - p_0 t) \tag{4.21}$$

expresses the Dirac electron-field operator, we obtain the standard form of the space operator:

$$\mathbf{X}(t) = \sum_s \int d\mathbf{p}\, b^{(+)*}(\mathbf{p}, s) \left[i\frac{\partial}{\partial \mathbf{p}} + t\frac{\mathbf{p}}{p_0} \right] b^{(+)}(\mathbf{p}, s) \tag{4.22}$$

only when the relation

$$u^{(+)*}(\mathbf{p}, s') \frac{\partial}{\partial \mathbf{p}} u^{(+)}(\mathbf{p}, s) = \frac{\mathbf{p}}{2m_0 p_0} \delta_{s's} \tag{4.23}$$

where $u^{(+)}(\mathbf{p}, s)$ expresses the positive energy spinor, is used. But this relation is valid only in the one-dimensional case. In this latter case the spinor $u^{(+)}(\mathbf{p}, s)$ has to be defined only by means of a special Lorentz transformation (see e.g. Bjorken & Drell, 1965).

Similarly, in the non-relativistic case we may analyse the conditions under which binary space-time compatibility in assured. Taking into consideration the one-dimensional case and defining the binary space operator as

$$X_1^{(b)}(t) = \int d\mathbf{p}\, a^{(+)*}(\mathbf{p}) \left[i\frac{\partial}{\partial p_1} + t\frac{p_1}{p_0} - i\frac{m_0^2}{2p_1 p_0^2} \right] a^{(+)}(\mathbf{p}) \tag{4.24}$$

we obtain

$$\begin{aligned} &\langle \Phi_1^{(+)} | [X_1^{(b)}(t), T_1(t)] | \Phi_1^{(+)} \rangle \\ &= m_0^2 \left\langle \frac{i}{p_0 p_1^2} \frac{\partial}{\partial p_1} \arg g^{(+)}(\mathbf{p}) - i\frac{t}{p_1 p_0^2} - \frac{1}{2p_1^3 p_0} + \frac{1}{2p_0^3 p_1} \right\rangle \end{aligned} \tag{4.25}$$

where $p_0 = \sqrt{(p_1^2 + m_0^2)}$. By performing the calculations, it may be proved that the compatibility of the binary space and binary time is assured when the time has a binary meaning with the imprecision given by $\langle m_0^2/2p_0 p_1^2 \rangle$. The imprecision thus obtained is identical with the imprecision operator average. The compatibility between the hermitian space operator $X_1(t)$ (binary space operator $X_1^{(b)}(t)$) and the binary time operator $T_1(t)$, (hermitian time operator $T_1^{(*)}(t)$) is assured if time is considered as a binary entity with imprecision

$$\left\langle \frac{p_0}{p_1^2} + \frac{1}{2p_0} \right\rangle, \qquad \left(\left\langle \frac{p_0}{2p_1^2} + \frac{1}{p_0} \right\rangle \right)$$

Using extended binary 'equivalence' it may be proved that the last imprecision is 'equivalent' to the real time imprecision $\langle m_0^2/2p_1^2 p_0 \rangle$ only when $\langle v_1 \rangle \leqslant c/2$, whereas the first imprecision is 'equivalent', for example to the imprecision $\langle (p_0/2p_1{}^2)+(1/4p_0)\rangle$. The existence of such an imprecision 'transfer' may be interpreted as an expression of the non-existence of a quantum-mechanical space-time symmetry.

5. *The Spherical Symmetric Approach to the Space-Time Quantisation*

We have proved that in order to unequivocally define the field-theoretic time for the *K–G*-field, the fulfilment of certain conditions is needed. As we shall see, the relevant peculiarities of the one-dimensional description may be reproduced in conditions under which the existence of a rotation symmetry of the field operators is assumed.

In these conditions the annihilation operator and the field operator may be expanded as spherical harmonics as follows:

$$a^{(+)}(\mathbf{p})=p^{-1}\sum_l Y_{l,0}\left(\frac{\mathbf{p}\cdot\mathbf{k}}{pk}\right)a_l^{(+)}(p) \tag{5.1}$$

and

$$\Phi^{(+)}(\mathbf{x},t)=r^{-1}\sum_l Y_{l,0}\left(\frac{\mathbf{k}\cdot\mathbf{x}}{kr}\right)\Phi_l^{(+)}(r,t) \tag{5.2}$$

respectively, where $\mathbf{k}/k$ is the unit vector in the direction of the symmetry axis and where $r\equiv|\mathbf{x}|$. Using the expansion

$$\exp i\mathbf{p}\cdot\mathbf{x}=\sum_l i^l\sqrt{4\pi(2l+1)}\,Y_{l,0}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{pr}\right)j_l(pr) \tag{5.3}$$

and the well-known properties of spherical harmonics one obtains the p-momentum representation of the *K–G*-field corresponding to a well-defined value of the angular momentum as:

$$\Phi_l^{(+)}(r,t)=\left(\frac{2}{\pi}\right)r\int_0^\infty \frac{p\,dp}{\sqrt{(2p_0)}}j_l(pr)\,a_l^{(+)}(p)\exp(-ip_0t) \tag{5.4}$$

where the commutation relation now takes the form

$$[a_l^{(+)}(p),a_{l'}^{(+)*}(p')]=\delta_{l'l}\,\delta(p'-p) \tag{5.5}$$

Defining, for the moment, the radial space operator by means of the relation

$$R^{(l)}(t)=i\int_0^\infty dr\,\Phi_l^{(+)*}(r,t)\,\overleftrightarrow{\partial}_t\,r\,\Phi_l^{(+)}(r,t) \tag{5.6}$$

and using the relation

$$\int_0^\infty dr\,r^3 j_l(p'r)j_l(pr)=\frac{1}{p}\frac{d}{dp}P\frac{1}{p^2-p'^2} \tag{5.7}$$

which is valid only for $l = 0$ and

$$\lim_{t\to+\infty} \frac{1}{p+p'} P \frac{1}{p-p'} \exp i(p_0' - p_0)\, t = -\, i\pi \frac{\delta(p'-p)}{2p} \tag{5.8}$$

we may conclude, having already performed the calculations, that the space operator may be adequately defined only at large t-values. Indeed

$$\mathscr{R}^{(l)}(t) \equiv \lim_{t\to+\infty} R^{(l)}(t) = \int_0^\infty dp\; a_l^{(+)*}(p) \left[i\frac{d}{dp} + t\frac{p}{p_0} \right] a_l^{(+)}(p) \tag{5.9}$$

where for the moment $l = 0$. But using the asymptotic field

$$\tilde{\Phi}_l^{(+)}(r,t) \equiv \lim_{t\to+\infty} \Phi_l^{(+)}(r,t) = \frac{-i}{\sqrt{(2\pi)}} \int_0^\infty \frac{dp}{\sqrt{(2p_0)}}\, a_l^{(+)}(p) \exp i(pr - p_0 t) \tag{5.10}$$

where only the dominant contribution has been maintained, and redefining the space operator as

$$\mathscr{R}^{(l)}(t) = i \int_0^\infty dr \tilde{\Phi}_l^{(+)*}(r,t)\; \overleftrightarrow{\partial}_t\, r\; \tilde{\Phi}_l^{(+)}(r,t) \tag{5.11}$$

it may be proved that expression (5.9) maintains its validity with respect to any value of the angular momentum. The space operator so obtained is an hermitian one when the annihilation operator $a_l^{(+)}(p)$ vanishes at zero and infinity.

It may also be easily proved that the field-theoretic counterpart

$$\mathscr{T}^{(l)}(t) = i \int_0^\infty dr\; \tilde{\Phi}_l^{(+)*}(r,t)\, r\hat{v}^{-1}\; \tilde{\Phi}_l^{(+)}(r,t) \tag{5.12}$$

of the quantum-mechanical time operator $r\hat{v}^{-1}$, where

$$\hat{v} = -\left(\frac{\partial}{\partial r}\right)\left(\frac{\partial}{\partial t}\right)^{-1} \tag{5.13}$$

expresses the velocity operator, is given by

$$\mathscr{T}^{(l)}(t) = \int_0^\infty dp\; a_l^{(+)*}(p) \left[i\frac{d}{dp_0} + t - i\frac{m_0^2}{p^2 p_0} \right] a_l^{(+)}(p) \tag{5.14}$$

when

$$\lim_{p\to 0} p^{-1/2} a_l^{(+)}(p) = 0, \qquad \lim_{p\to\infty} a_l^{(+)}(p) = 0 \tag{5.15}$$

where the weak limit has been considered.

The above conditions also expressed the requirements needed in order to define the hermitian time operator

$$\mathscr{T}^{(l)(*)}(t) = \int_0^\infty dp\; a_l^{(+)*}(p) \left[i\frac{d}{dp_0} + t - i\frac{m_0^2}{2p^2 p_0} \right] a_l^{(+)}(p) \tag{5.16}$$

If we were trying to define space with respect to time, then unsatisfactory dynamical expressions would be obtained. Indeed the field-theoretical counterpart of the presumptive quantum-mechanical 'space' operator $t\hat{v}$ takes the form

$$\int_0^\infty dp\, a_l^{(+)*}(p)\left[t\frac{p}{p_0}+i\frac{p}{2p_0^2}-i\frac{m_0^2}{pp_0^2}\right]a_l^{(+)}(p) \tag{5.17}$$

which only apparently possesses the dynamical meaning of a real space operator.

Consequently if time may be defined in respect with space, the converse way of defining space with respect to time generally is not possible. For this purpose the existence of a time representation placed on the same footing as the space coordinate representation would be required. If in the non-relativistic case such a representation could be defined (see e.g. Gien, 1969), there arise, appreciable difficulties in defining a time-representation within relativistic quantum-field theories.

Similarly to the one-dimensional case, we may define the binary space operator as

$$\mathscr{R}_l^{(b)}(t)=\int_0^\infty dp\, a_l^{(+)*}(p)\left[i\frac{d}{dp}+t\frac{p}{p_0}-i\frac{m_0^2}{2pp_0^2}\right]a_l^{(+)}(p) \tag{5.18}$$

Starting from

$$(\Phi_1^{(+)(l)}|[\mathscr{R}_l^{(b)}(t),\mathscr{T}^{(l)}(t)]|\Phi_1^{(+)(l)})$$
$$=m_0^2\left\langle\frac{i}{p^2p_0}\frac{d}{dp}\arg g_l^{(+)}(p)-i\frac{t}{pp_0^2}-\frac{m_0^2}{2p^3p_0^3}\right\rangle_l \tag{5.19}$$

where the single-particle state is now given by

$$|\Phi_1^{(+)(l)})=\int_0^\infty dp\, g_l^{(+)}(p)\, a_l^{(+)*}(p)|0\rangle \tag{5.20}$$

it may be proved that binary space-time compatibility is assured under the conditions in which time is considered as a binary entity with imprecision given by $\langle m_0^2/2p^2p_0\rangle_l$. As expected, this imprecision is identical with the imprecision operator average and also with the real time imprecision previously obtained (Papp, 1972).

Summarising the results of Sections 4 and 5 we may conclude that, for the *K–G*-field, the present time quantisation may be consistently and unequivocally performed only with respect to the one-dimensional cases or under the conditions in which the p-momentum representation of the *K–G*-field may be defined.

6. *Action and Space-Time Quantisation*

In order to prove the mutual connection between the space-time quantisation previously performed and the action quantisation we shall firstly define the binary action operators. It may be remarked that the binary action description expresses in fact—in a more extended sense—the usual peculiarities of the action quantisation. Finally we obtain the result that we cannot attribute a measurable meaning to the action without attributing to time (up to binary 'equivalence') the previously defined binary meaning and vice versa.

Calculating in the non-relativistic case the field-theoretic counterparts of the classical actions pr, $pr - \omega t$ and ωt we obtain that the field-theoretical binary action is—up to the factor $\frac{1}{2}$—unequivocally defined. Thus the field-theoretic counterpart of the quantum-mechanical action $r\hat{p}$, where $\hat{p} = -i(d/dr)$, is given by

$$A^{(l)}(t) = \int_0^\infty dp\, a_l^*(p) \left[ip\frac{d}{dp} + t\frac{p^2}{2m_0} + i \right] a_l(p), \qquad (t \to +\infty) \tag{6.1}$$

where the p-momentum representation of the S-field has been used. In these conditions it may be easily proved that the action imprecision is given by universal constant $\hbar/2$. On the other hand we may consider, in agreement with the basic assumptions of quantum mechanics, that $\hbar/2$ expresses the minimum 'observable' value of the action encountered within the multitude of quantum-mechanical objects. Considering the role of the measuring apparatus, we may assume that the action values smaller than $\hbar/2$ lose any observable meaning. Thus, the physical meaning of the action quantisation agrees with that of the binary formalism. In this way we are able to legitimise the binary interpretative formalism as a general quantisation-methodology.

The average value of the non-relativistic action is given by

$$(\psi_1^{(l)}|A^{(l)}(t)|\psi_1^{(l)}) = \left\langle -p\frac{d}{dp}\arg g_l(p) + t\frac{p^2}{2m_0} + \frac{i}{2} \right\rangle_l \tag{6.2}$$

where the single-particle state

$$|\psi_1^{(l)}) = \int_0^\infty dp\, g_l(p)\, a_l^*(p)|0\rangle \tag{6.3}$$

has been normalised to unity. Consequently the non-relativistic action possesses a measurable meaning when time possesses a binary meaning with the imprecision given by $\langle 1/4\omega \rangle_l$ (Papp, 1971). Thus, time quantisation is able to assure a measureable meaning of the action and vice versa.

In the relativistic case the uniqueness of the field-theoretic action operator ceases to be preserved. Starting from the classical actions pr, $pr - p_0t$ and p_0t we may define the corresponding quantum-mechanical operators as

$$\hat{a}_1 = ip\frac{d}{dp} \tag{6.4}$$

$$\hat{a}_{12} = -i\frac{m_0^2}{p}\frac{d}{dp} \tag{6.5}$$

and

$$\hat{a}_2 = i\frac{p_0^2}{p}\frac{d}{dp} \tag{6.6}$$

respectively, where the p-momentum representation has been used. The field-theoretical counterparts of the above operators are given (for $t \to +\infty$) by

$$A_1^{(l)*}(t) = \int_0^\infty dp\, a_l^{(+)*}(p)\left[ip\frac{d}{dp} + t\frac{p^2}{p_0}\right]a_l^{(+)}(p) \tag{6.7}$$

$$A_{12}^{(l)*}(t) = \int_0^\infty dp a_l^{(+)*}(p)\left[-i\frac{m_0^2}{p}\frac{d}{dp} - t\frac{m_0^2}{p_0}\right]a_l^{(+)}(p) \tag{6.8}$$

and

$$A_{12}^{(l)*}(t) = \int_0^\infty dp\, a_l^{(+)*}(p)\left[i\frac{p_0^2}{p}\frac{d}{dp} + p_0 t\right]a_l^{(+)}(p) \tag{6.9}$$

respectively. Consequently

$$(\Phi_1^{(+)(l)}|A_1^{(l)*}(t)|\Phi_1^{(+)(l)}) = \left\langle -p\frac{d}{dp}\arg g_l^{(+)}(p) + t\frac{p^2}{p_0} - \frac{i}{2}\right\rangle_l \tag{6.10}$$

$$(\Phi_1^{(+)(l)}|A_{12}^{(l)*}(t)|\Phi_1^{(+)(l)}) = \left\langle -\frac{m_0^2}{p}\frac{d}{dp}\arg g_l^{(+)}(p) - t\frac{m_0^2}{p_0} - i\frac{m_0^2}{2p^2}\right\rangle_l \tag{6.11}$$

and

$$(\Phi_1^{(+)(l)}|A_2^{(l)*}(t)|\Phi_1^{(+)(l)}) = \left\langle -\frac{p_0^2}{p}\frac{d}{dp}\arg g_l^{(+)}(p) + p_0 t + i\frac{m_0^2 - p^2}{2p^2}\right\rangle_l \tag{6.12}$$

Under these conditions, using the previously defined methodology, the action imprecisions may be defined as

$$\frac{1}{2}, \quad \left\langle\frac{m_0^2}{2p^2}\right\rangle_l \quad \text{and} \quad \left|\left\langle\frac{m_0^2 - p^2}{2p^2}\right\rangle_l\right| \tag{6.13}$$

respectively, so that the implied time imprecisions become

$$\left\langle\frac{p_0}{2p^2}\right\rangle_l, \quad \left\langle\frac{p_0}{2p^2}\right\rangle_l \quad \text{and} \quad \left|\left\langle\frac{m_0^2 - p^2}{2p^2 p_0}\right\rangle_l\right| \tag{6.14}$$

respectively. Therefore, in the relativistic case, only the pr-action operator possesses the usual imprecision $\hbar/2$. In spite of the heterogenous form of the action imprecisions, previously calculated the total time quantum $\langle p_0/2p^2\rangle_l$, preserves its uniqueness with respect to the pr and $(pr - p_0 t)$ actions. However, some aspects of the meaning and the role of the $p_0 t$ action imprecision, need additional investigation.

Consequently, if in the nonrelativistic case the space-time quantisation and the action quantisation may be straightforwardly connected, in the relativistic case—besides the presence of some additional difficulties—such a connection cannot be established without using extended binary 'equivalence.'

7. *Conclusions*

In this paper some methods assuring the possibility of performing a suitable binary space-time description within quantum field theory have been analysed and clarified. In this way a more profound understanding of the peculiarities of the space-time description was acquired. We can remark that space and time cannot be placed on the same footing. This fact itself could be able to affect some usual aspects attributed to the relativistic four-dimensional space-time world. On the other hand not all the inertial reference frames are "microscopically" equivalent. Indeed, in order to perform the present space-time quantisation the proper-system was excluded (together with a certain 'vicinity' of frames in which the momentum takes very small values) and the reference frames in which the momentum takes very large values were also excluded. However, the compatibility between the present formulation of the space-time quantisation and the general requirements of Lorentz covariance is "microscopically" preserved, because it is the proper-time which is measured in respect to the proper-system. On the other hand, if we assume that the measuring apparatus includes a reference frame, some inconsistencies connected with the standard requirements of Lorentz covariance would be implied. Indeed, passing from one reference frame to another one, several apparata are implied and therefore—because the imprecisions previously defined are not Lorentz invariant—they are apparata with several imprecisions. In this respect the aim to include results obtained by using several apparata in the same 'equivalence' class is submitted to certain limitations at least under the conditions in which the implied imprecisions take appreciable different values. We may also remark that, in contrast to the invariance of the action $p_1 x_1 - p_0 t$ with respect to the special Lorentz transformation, the 'quantisation' of the above action leads to the imprecision $\langle m_0^2/2p_1^2 \rangle$, which is not a Lorentz-invariant quantity.

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